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SPECIAL ISSUE

On Optimal Control Problems with Control in a Disc

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Abstract—We consider a time-optimal control problem with Fuller-type symmetry and with control in the 2-dimensional unit disk. The problem can be solved analytically, with an implicit representation of the Bellman function. The optimal value of this problem serves as an upper bound on the optimal value of another optimal control problem with Fuller-type symmetry and with a second-order singular trajectory, which cannot be solved analytically.

Keywords: optimal control, Fuller-type symmetry, singular trajectories, time-optimal control

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1. INTRODUCTION

Optimal control problems are usually solved by means of the Pontryagin Maximum Principle (PMP), which leads to a Hamiltonian system with discontinuous right-hand side [7]. Since the Lipschitz-condition, as a consequence of the discontinuity, does not hold, the theorem of existence and uniqueness of solutions to the Ordinary Differential Equation (ODE) does not hold, and the phase portrait can exhibit various kinds of singularities, which appear in the optimal synthesis as singular trajectories. Since such singular trajectories often arise as part of the optimal synthesis, it is necessary to have a good understanding of these singularities. To this end a model problem is considered which on the one hand, is easy enough to be solved, in the best case analytically, and on the other hand, exhibits the kind of singularity under study.

The phenomenon of singular trajectories in optimal control was first discovered in [1], where an example of a system was exhibited where the optimal control performs an infinite number of switchings in finite time, so-called *chattering*. Singular trajectories were studied systematically in, e.g., [2–4, 6]. The phenomenon of chattering was studied in detail in [5, 8]. A more complicated system exhibiting chattering combined with a fractal optimal control pattern was investigated in [9, 10].

In this work we study singular trajectories of second order, continuing the research program initiated in the seminal work [8]. The first optimal control problem with a second-order singular trajectory has been solved in [1]. It is given by the formulation

$$\min \int_0^\infty \frac{x^2}{2}\,dt : \qquad \dot{x} = y, \quad \dot{y} = u, \quad u \in [-1, 1]. \tag{1}$$

The second-order singular trajectory present in the phase portrait of the corresponding Hamiltonian system is the trajectory $x(t) = y(t) \equiv 0$. A junction of a generic optimal trajectory of problem (1) with the singular one is performed in finite time with an infinite number of switchings of the control u between the extremal values ± 1. The Bellman function of problem (1) has been explicitly computed in [9] and is given by

$$\omega_{1D}(x,y) = \begin{cases} -\frac{1}{2}x^2y - \frac{1}{3}xy^3 - \frac{1}{15}y^5 - \gamma(y^2+2x)^{\frac{5}{2}}, & x \geqslant -\beta y|y|; \\ \frac{1}{2}x^2y - \frac{1}{3}xy^3 + \frac{1}{15}y^5 - \gamma(y^2-2x)^{\frac{5}{2}}, & x \leqslant -\beta y|y|, \end{cases} \tag{2}$$

where $\beta \approx 0.4446$ solves the equation

$$36\beta^4 + 3\beta^2 - 2 = 0$$

and $\gamma = \frac{-\beta^2+2\beta-\frac{2}{3}}{10(1-2\beta)^{\frac{3}{2}}} \approx 0.06753$. Recall that the expression $-\omega_{1D}(x_0, y_0)$ is the optimal value of the problem with initial data $x(0) = x_0$, $y(0) = y_0$.

The optimal synthesis of the problem exhibits a continuous symmetry. Namely, if $(x(t), y(t), u(t))$ is an optimal solution of problem (1), then for every $\lambda > 0$ the trajectory

$$(x_\lambda(t), y_\lambda(t), u_\lambda(t)) = (\lambda^2 x(\lambda^{-1}t), \lambda y(\lambda^{-1}t), u(\lambda^{-1}t)) \tag{3}$$

is also optimal [8]. Similarly, for every $\lambda > 0$ the Bellman function obeys the relation

$$\omega_{1D}(\lambda^2 x, \lambda y) = \lambda^5 \omega_{1D}(x,y). \tag{4}$$

In [8] an analog of problem (1) with two-dimensional control was considered, namely

$$\min \int_0^\infty \frac{\|x\|^2}{2}\,dt: \qquad \dot{x} = y, \quad \dot{y} = u, \quad u \in U = \mathbb{D}, \tag{5}$$

where $\mathbb{D} = \{u \in \mathbb{R}^2 \mid \|u\| \leqslant 1\}$ is the unit disk. Here $x(t)$, $y(t)$ are vector-valued functions. This problem also features a second-order singular trajectory at the origin $(x,y) = (0,0)$ of the space $\mathbb{R}^2 \times \mathbb{R}^2$. It exhibits the same symmetry (3), but also an additional rotational symmetry. Namely, for every optimal solution $(x(t), y(t), u(t))$ of problem (5) and every orthogonal matrix $O \in O(2)$ the trajectory

$$(Ox(t), Oy(t), Ou(t)) \tag{6}$$

is also optimal. The Bellman function ω_{2D} of problem (5) satisfies the symmetries

$$\omega_{2D}(\lambda^2 x, \lambda y) = \lambda^5 \omega_{2D}(x,y), \quad \omega_{2D}(Ox, Oy) = \omega_{2D}(x,y)$$

for every $\lambda > 0$ and every $O \in O(2)$. The rotational symmetry implies that the dynamics of the optimal synthesis factors through to the Gramian $\begin{pmatrix} \langle x,x\rangle & \langle x,y\rangle \\ \langle x,y\rangle & \langle y,y\rangle \end{pmatrix}$ of the vectors x, y. The value at time t of this 2×2 matrix is determined solely by the initial value of the Gramian itself.

It was shown in [8, Proposition 7.8] that for linearly dependent initial vectors $x(0), y(0)$ problem (5) reduces to problem (1), and the vectors $x(t), y(t)$ stay in the 1-dimensional subspace spanned by the initial vectors for all time. In particular, for linearly dependent vectors $x = r_x \cdot (\cos\varphi, \sin\varphi)^T$, $y = r_y \cdot (\cos\varphi, \sin\varphi)^T$ the Bellman function satisfies

$$\omega_{2D}(x,y) = \omega_{1D}(r_x, r_y),$$

where r_x, r_y are allowed to take arbitrary real values.

Besides the optimal trajectories of problem (5) emanating from linearly dependent initial values, a family of *self-similar* optimal trajectories has been computed in [8, Proposition 7.9, Corollary 7.3].

For these trajectories, the tangent of the angle between the vectors x and y is constant and equals $-\sqrt{5}/2$ (the values $\sqrt{5}$ and $\sqrt{5/2}$ in [8, p.233] are both erroneous), and between the vectors y and u it is $-\sqrt{5}$. Moreover, $2\langle y,y\rangle = \sqrt{6\langle x,x\rangle}$. It follows that the Gramian of the initial values $x(0), y(0)$ is given by

$$\begin{pmatrix} \langle x(0),x(0)\rangle & \langle x(0),y(0)\rangle \\ \langle x(0),y(0)\rangle & \langle y(0),y(0)\rangle \end{pmatrix} = \begin{pmatrix} \frac{\lambda_0^4}{54} & -\frac{\lambda_0^3}{27} \\ -\frac{\lambda_0^3}{27} & \frac{\lambda_0^2}{6} \end{pmatrix} \tag{7}$$

for some $\lambda_0 > 0$. The Gramian of the corresponding trajectory evolves according to the formula

$$\begin{pmatrix} \langle x(t),x(t)\rangle & \langle x(t),y(t)\rangle \\ \langle x(t),y(t)\rangle & \langle y(t),y(t)\rangle \end{pmatrix} = \begin{pmatrix} \frac{\lambda(t)^4}{54} & -\frac{\lambda(t)^3}{27} \\ -\frac{\lambda(t)^3}{27} & \frac{\lambda(t)^2}{6} \end{pmatrix}, \quad \lambda(t) = \lambda_0 - t. \tag{8}$$

It follows that the parameter λ_0 is the arrival time at the singular trajectory, which is located at the origin $(x, y) = (0, 0)$.

While the angles between the vectors x, y, u remain constant, the vectors themselves revolve ever more rapidly around the origin, making an infinite number of revolutions in finite time. More concretely, the direction each vector is pointing is given by the time-varying angle [8, Proposition 7.9, Corollary 7.3]

$$\varphi(t) = \sigma\sqrt{5}\log(\lambda_0 - t) + \text{const}, \tag{9}$$

where $\sigma \in \{-1, +1\}$ determines the direction of revolution and the additive constant on the initial conditions.

The complete optimal synthesis of problem (5) is currently unknown. In this paper we compute an upper bound on the objective value by constructing a sub-optimal solution. This upper bound has to satisfy several, potentially conflicting criteria:

- the bound should be reasonable close to the true value
- the bound should be efficiently usable numerically, e.g., given by a global analytic expression

As we have seen from the analysis above, the optimal solutions for different initial values can be quite different from each other. Constructing a bound which is everywhere close to the optimal value and at the same time not given by a multitude of different expressions for different phase space regions is a challenging task.

We cope with this difficulty by solving the *time-optimal* control problem

$$\min\,(T - t_0): \quad \dot{x} = y, \quad \dot{y} = u, \quad u \in U = \mathbb{D}, \quad x(T) = y(T) = 0, \quad x(t_0) = x_0, \quad y(t_0) = y_0, \tag{10}$$

which up to a shift of the time variable t has the same feasible set of trajectories as problem (5) but another objective value. This is accomplished in Section 2. In Section 3 we substitute the obtained time-optimal solution in the objective value of the original problem (5) to obtain the upper bound. Finally, in Section 4 we compare the upper bound with the optimal value of problem (5) on those trajectories where the latter is known. It turns out that, on the one hand, the quality of the bound is reasonably good on all initial values for which the optimal solution of problem (5) is known, and on the other hand it is given by a single analytic expression.

2. TIME-OPTIMAL CONTROL PROBLEM

In this section we analytically solve the time-optimal control problem (10).

Let us apply the PMP. Introduce adjoint variables ϕ, ψ and assemble the Pontryagin function

$$\mathcal{H} = -1 + \langle \phi, y \rangle + \langle \psi, u \rangle. \tag{11}$$

The optimal control is then given by $\hat{u} = \arg\max_{u \in U} \mathcal{H} = \frac{\psi}{\|\psi\|}$ whenever $\psi \neq 0$. The dynamics of the adjoint variables is given by

$$\dot{\psi} = -\frac{\partial \mathcal{H}}{\partial y} = -\phi, \quad \dot{\phi} = -\frac{\partial \mathcal{H}}{\partial x} = 0.$$

Since the terminal time instant T is not fixed, we also have the transversality condition

$$\mathcal{H}(T) = \langle \psi, \hat{u} \rangle - 1 = \|\psi\| - 1 = 0. \tag{12}$$

Hence the function $\psi(t) = \phi t + \psi(0)$ is affine and at $t = T$ terminates on the unit circle.

By virtue of the rotational symmetry, without loss of generality we may assume that $\dot{\psi} = -\phi = (\alpha, 0)^T$ is collinear with the unit basis vector $e_1 = (1,0)^T$ and $\alpha \geqslant 0$. In the case $\alpha > 0$ we shall choose the initial value t_0 of the time variable such that $\psi(0) = (0, \beta)^T$ is collinear with e_2, and $\beta \geqslant 0$.

Case $\alpha\beta = 0$: In this case the adjoint variable ψ evolves in a 1-dimensional linear subspace of $\mathbb{R}^2$. Hence also u, y, x have to evolve in this subspace, and the problem reduces to the well-known 1-dimensional time-optimal control problem with acceleration bounded by 1, which is given by (10) with all variables considered as scalars.

In this case the adjoint variable ϕ is a scalar constant. Let us shift the time variable such that the final time T is zero. Then the adjoint variable ψ has terminal value $\psi(T) = \psi(0) = \sigma \in \{-1, +1\}$ and is given by $\psi(t) = \sigma - \phi t$. The optimal control u is piece-wise constant, given by

$$\hat{u}(t) = \begin{cases} +1, & \text{if } \sigma > \phi t, \\ -1, & \text{if } \sigma < \phi t. \end{cases}$$

Equivalently, with $\tilde{\phi} = \sigma\phi$ we get

$$\hat{u}(t) = \begin{cases} +\sigma, & \text{if } 1 > \tilde{\phi} t, \\ -\sigma, & \text{if } 1 < \tilde{\phi} t. \end{cases}$$

This yields

$$y(t) = \int_0^t \hat{u}(s)\,ds = \begin{cases} \sigma t, & \text{if } 1 > \tilde{\phi} t, \\ -\sigma(t - \tilde{\phi}^{-1}) + \sigma\tilde{\phi}^{-1}, & \text{if } 1 < \tilde{\phi} t \end{cases}$$

and further

$$x(t) = \int_0^t y(s)\,ds = \begin{cases} \sigma\dfrac{t^2}{2}, & \text{if } 1 > \tilde{\phi} t, \\ -\sigma\dfrac{t^2 - \tilde{\phi}^{-2}}{2} + 2\sigma\tilde{\phi}^{-1}(t - \tilde{\phi}^{-1}) + \sigma\dfrac{\tilde{\phi}^{-2}}{2}, & \text{if } 1 < \tilde{\phi} t \end{cases}$$

The case $1 > \tilde{\phi} t$ is hence possible only if $x = \sigma\frac{y^2}{2} = -\frac{y|y|}{2}$, in which case $t_0 = \sigma y_0$. For any other point in the x, y plane we must have $1 < \tilde{\phi} t$. Hence the locus of the equation $x + \frac{y|y|}{2} = 0$ separates the x, y plane in two regions, where different values of the control must be optimal. The separating curve consists of the two trajectories which arrive directly at the origin (see Fig. 1).

In particular, for $1 < \tilde{\phi} t$ we have

$$\hat{u} = -\sigma = -\operatorname{sgn}\left(x_0 + \frac{y_0|y_0|}{2}\right).$$

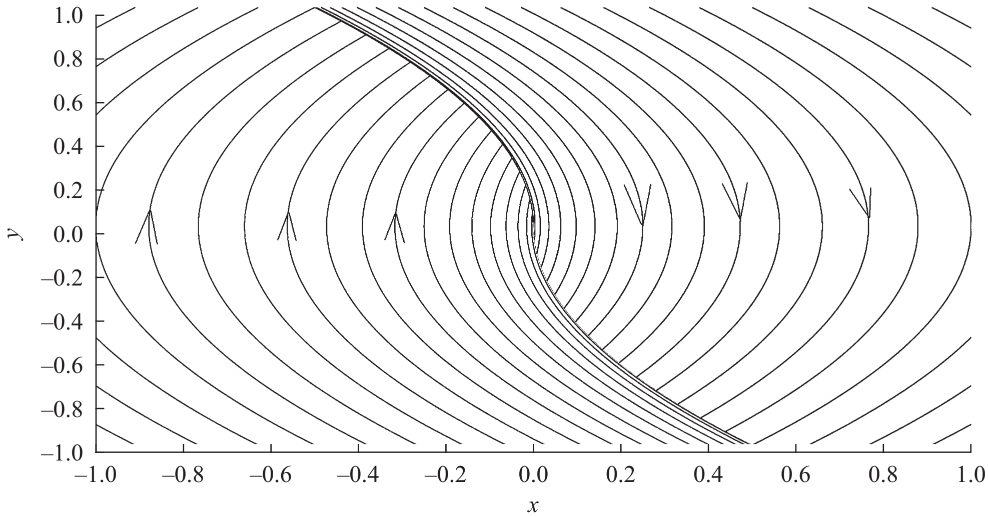

Fig. 1. Optimal synthesis for the 1-dimensional time-optimal control problem. To the left of the curve separating the two regions we have $\hat{u} = +1$ and $\sigma = -1$, to the right $\hat{u} = -1$ and $\sigma = 1$. The trajectories of the system switch to the opposite control when they arrive at the separating curve.

Further, $\sigma y_0 = -t_0 + 2\tilde{\phi}^{-1}$, $\sigma x_0 = -\frac{t_0^2}{2} + 2\tilde{\phi}^{-1}t_0 - \tilde{\phi}^{-2}$ and hence

$$\tilde{\phi}^{-1} = -\sqrt{\sigma x_0 + \frac{y_0^2}{2}}, \quad t_0 = -\sigma y_0 - 2\sqrt{\sigma x_0 + \frac{y_0^2}{2}}.$$

For $t_0 < t < -\sqrt{\sigma x_0 + \frac{y_0^2}{2}}$ we thus finally obtain

$$y(t) = -\sigma t - 2\sigma\sqrt{\sigma x_0 + \frac{y_0^2}{2}}, \quad x(t) = -\frac{\sigma t^2}{2} - 2\sigma\sqrt{\sigma x_0 + \frac{y_0^2}{2}}t - x_0 - \frac{\sigma y_0^2}{2}$$

with $\sigma = \operatorname{sgn}\left(x_0 + \frac{y_0|y_0|}{2}\right)$.

Case $\alpha > 0$, $\beta > 0$: In this case $\psi(t) = (\alpha t, \beta)^T$. Since $\|\psi(T)\| = 1$, we must have $\beta \leqslant 1$ and $\alpha^2 T^2 + \beta^2 = 1$.

Introduce the scaled time variable $\tau = \alpha t$ and the scaled starting point $\tau_0 = \alpha t_0$ and end-point $\bar{\tau} = \alpha T$. Then we get $\beta = \sqrt{1 - \bar{\tau}^2}$. Consequently, $\|\psi(t)\| = \sqrt{\tau^2 + 1 - \bar{\tau}^2}$. The optimal control is then given by

$$\hat{u} = \frac{\psi}{\|\psi\|} = \frac{(\tau, \beta)^T}{\sqrt{\tau^2 + \beta^2}}. \tag{13}$$

Since the system is autonomous, the Pontryagin function (11) is constant along the trajectory, and by the transversality condition (12) we get the energy integral $\mathcal{H} = -1 - \alpha y_1 + \sqrt{\tau^2 + \beta^2} \equiv 0$. Hence

$$y_1 = \frac{\sqrt{\tau^2 + \beta^2} - 1}{\alpha}. \tag{14}$$

For the other component of $y(t)$ we get the solution

$$y_2 = \int\limits_T^t \frac{\beta}{\sqrt{\alpha^2 s^2 + \beta^2}}\, ds = \int\limits_{\bar{\tau}}^{\tau} \frac{\beta}{\sqrt{s^2 + \beta^2}} \frac{ds}{\alpha} = \frac{\beta}{\alpha}\left(\operatorname{arsinh}\frac{\tau}{\beta} - \operatorname{artanh}\bar{\tau}\right). \tag{15}$$

Here we used that $\operatorname{arsinh}\frac{\bar{\tau}}{\beta} = \operatorname{arsinh}\frac{\bar{\tau}}{\sqrt{1-\bar{\tau}^2}} = \operatorname{artanh}\bar{\tau}$.

Integrating further, we obtain for the function $x(t) = \int_T^t y(s)\,ds = \int_\tau^{\bar\tau} y(s/\alpha)\,\frac{ds}{\alpha}$ that

$$x_1 = \frac{1}{2\alpha^2}\tau\sqrt{\tau^2+\beta^2} + \frac{\beta^2}{2\alpha^2}\operatorname{arsinh}\frac{\tau}{\beta} - \frac{\tau}{\alpha^2} - \frac{1}{2\alpha^2}(\beta^2\operatorname{artanh}\bar\tau - \bar\tau),$$
$$x_2 = \frac{\beta\tau}{\alpha^2}\operatorname{arsinh}\frac{\tau}{\beta} - \frac{\beta}{\alpha^2}\sqrt{\tau^2+\beta^2} - \frac{\beta\tau}{\alpha^2}\operatorname{artanh}\bar\tau + \frac{\beta}{\alpha^2}.$$

Let us compute the elements of the Gramian. After insertion into the scalar products and simplification we get

$$\begin{aligned}
\alpha^4\|x\|^2 &= \frac14\tau^4 + \left(\beta^2\operatorname{artanh}^2\bar\tau + \frac{5\beta^2}{4} + 1\right)\tau^2 - \left(\beta^2\operatorname{artanh}\bar\tau + \bar\tau\right)\tau \\
&* + \frac14\left(4\beta^4 + 3\beta^2 + 1 + \beta^4\operatorname{artanh}^2\bar\tau - 2\beta^2\bar\tau\operatorname{artanh}\bar\tau\right) \\
&* - \tau^2\sqrt{\tau^2+\beta^2} + \frac12\left(3\beta^2\operatorname{artanh}\bar\tau + \bar\tau\right)\tau\sqrt{\tau^2+\beta^2} - 2\beta^2\sqrt{\tau^2+\beta^2} \\
&* - 2\beta^2\tau^2\operatorname{artanh}\bar\tau\operatorname{arsinh}\frac{\tau}{\beta} + \beta^2\tau\operatorname{arsinh}\frac{\tau}{\beta} - \frac{\beta^2}{2}\left(\beta^2\operatorname{artanh}\bar\tau - \bar\tau\right)\operatorname{arsinh}\frac{\tau}{\beta} \\
&* - \frac{3\beta^2}{2}\tau\sqrt{\tau^2+\beta^2}\operatorname{arsinh}\frac{\tau}{\beta} + \beta^2\tau^2\operatorname{arsinh}^2\frac{\tau}{\beta} + \frac{\beta^4}{4}\operatorname{arsinh}^2\frac{\tau}{\beta},
\end{aligned} \tag{16}$$

$$\begin{aligned}
*\alpha^3\langle x,y\rangle &= \frac12\tau^3 + \left(\beta^2\operatorname{artanh}^2\bar\tau + \frac{\beta^2}{2} + 1\right)\tau - \frac{\beta^2}{2}\sqrt{\tau^2+\beta^2}\operatorname{arsinh}\frac{\tau}{\beta} \\
&* + \frac12\left(\beta^2\operatorname{artanh}\bar\tau + \bar\tau\right)\sqrt{\tau^2+\beta^2} - \frac32\tau\sqrt{\tau^2+\beta^2} + \frac{\beta^2}{2}\operatorname{arsinh}\frac{\tau}{\beta} - \frac12\left(\beta^2\operatorname{artanh}\bar\tau + \bar\tau\right) \\
&* + \beta^2\tau\operatorname{arsinh}^2\frac{\tau}{\beta} - 2\beta^2\tau\operatorname{artanh}\bar\tau\operatorname{arsinh}\frac{\tau}{\beta},
\end{aligned} \tag{17}$$

$$*\alpha^2\|y\|^2 = \tau^2 - 2\sqrt{\tau^2+\beta^2} + \beta^2\operatorname{arsinh}^2\frac{\tau}{\beta} - 2\beta^2\operatorname{artanh}\bar\tau\operatorname{arsinh}\frac{\tau}{\beta} + \left(\beta^2\operatorname{artanh}^2\bar\tau + \beta^2 + 1\right). \tag{18}$$

2.1. Computation of the Parameters $\alpha, \bar\tau, \tau_0$

In order to compute the time-optimal trajectory for a given initial value of the Gramian $\begin{pmatrix} \|x_0\|^2 & \langle x_0,y_0\rangle \\ \langle x_0,y_0\rangle & \|y_0\|^2 \end{pmatrix}$, we have to invert the above dependence to obtain the values $\alpha, \bar\tau, \tau_0$.

The dependence on α is algebraic. Multiplication of α by a constant λ multiplies the Gramian from the left and from the right by the diagonal matrix $\operatorname{diag}(\lambda^{-2}, \lambda^{-1})$. In the cone $\mathcal{S}_+^2$ of positive semi-definite 2×2 matrices this action defines 1-dimensional orbits of radial type, each of which intersects every affine compact non-zero section of the cone $\mathcal{S}_+^2$ in exactly one point. The orbit itself then depends only on the parameters $\tau_0, \bar\tau$. It can be represented, e.g., by the two ratios $\frac{\|x_0\|^2}{\|y_0\|^4}$, $\frac{\langle x_0,y_0\rangle}{\|y_0\|^3}$.

The dependence of the orbit on the parameters cannot be inverted in closed form. In order to shed light on it, let us compute the limit when the parameters $\tau_0, \bar\tau$ tend to the boundary of their domain of definition. Recall that $-1 < \bar\tau < 1$, $\tau_0 < \bar\tau$.

In the limit $\tau_0 \to \bar\tau$ we obtain $\operatorname{arsinh}\frac{\tau_0}{\beta} \to \operatorname{arsinh}\frac{\bar\tau}{\beta} = \operatorname{artanh}\bar\tau$, $\sqrt{\tau_0^2+\beta^2} \to 1$. Inserting with $\tau = \tau_0$ into (16), (17), (18) we obtain that the Gramian of the initial point (x_0, y_0) tends to 0. However, if at the same time $\alpha \to 0$ such that the ratio $\frac{\bar\tau-\tau_0}{\alpha} = T - t_0$ equals 1, then the control

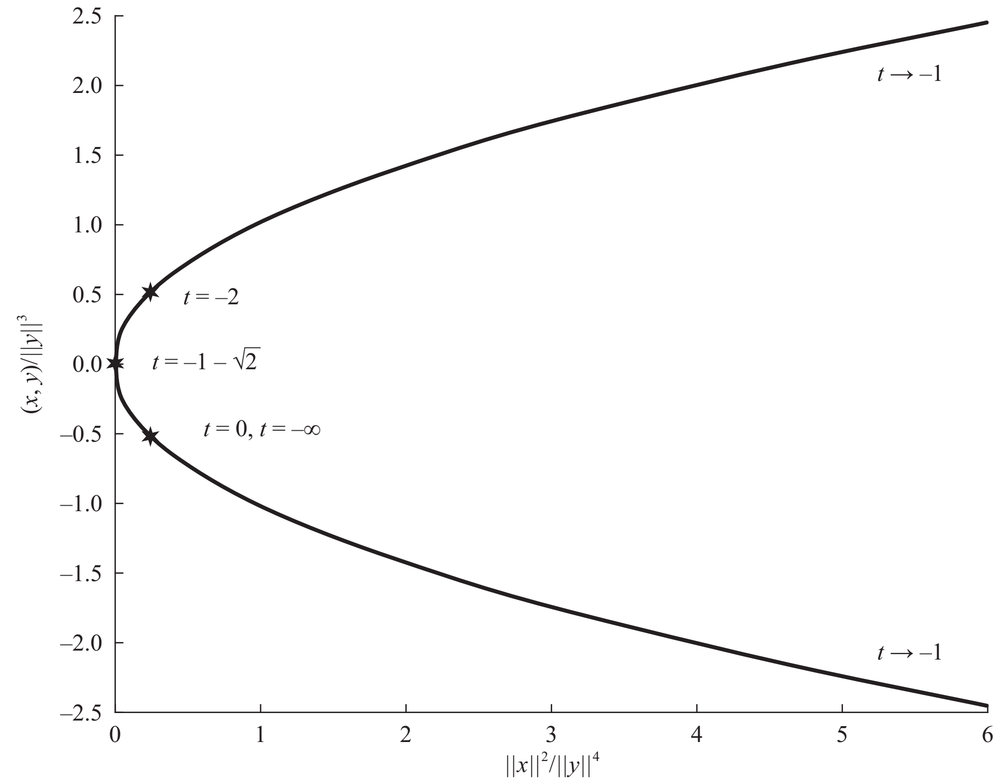

Fig. 2. Limits of the ratios $\frac{\|x_0\|^2}{\|y_0\|^4}$, $\frac{\langle x_0,y_0\rangle}{\|y_0\|^3}$ when $\bar\tau, \tau_0$ tend to the boundary of their domain of definition. The limit is different from $(\frac{1}{4}, -\frac{1}{2})$ only for $\bar\tau \to 1$, $\tau_0 < 0$ and is located on a parabola.

tends to the constant function $\hat u \equiv (\bar\tau, \beta)^T$ and the trajectory tends to a segment of a parabola given by $x(t) = \frac{1}{2}(t-T)^2\hat u$, $y(t) = (t-T)\hat u$. Hence $x_0 \to \frac{1}{2}(\bar\tau,\beta)^T$, $y_0 \to -(\bar\tau,\beta)^T$, and the Gramian tends to $\begin{pmatrix} \frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}$. In particular, $\frac{\|x_0\|^2}{\|y_0\|^4} \to \frac{1}{4}$, $\frac{\langle x_0,y_0\rangle}{\|y_0\|^3} \to -\frac{1}{2}$.

In the limit $\bar\tau \to \pm 1$ we get $\beta \to 0$, $\beta \operatorname{arsinh}\frac{\tau}{\beta} \to 0$, $\beta \operatorname{artanh}\bar\tau \to 0$, $\sqrt{\tau^2+\beta^2} \to |\tau|$. Inserting into (16), (17), (18) we get

$$\alpha^4\|x\|^2 \to \frac{1}{4}\tau^4 + \tau^2 + \frac{1}{4} - \tau^2|\tau| + \frac{1}{2}\bar\tau\tau|\tau| - \bar\tau\tau,$$

$$\alpha^3\langle x,y\rangle \to \frac{1}{2}(|\tau|-1)\left(\tau(|\tau|-2)+\bar\tau\right),$$

$$\alpha^2\|y\|^2 \to (|\tau|-1)^2.$$

For $\bar\tau \to -1$ we have $\tau \leqslant -1$ and $|\tau| = -\tau$. Setting $\alpha = -1-\tau_0$ such that again $T - t_0 \to 1$, this yields $\begin{pmatrix} \|x_0\|^2 & \langle x_0,y_0\rangle \\ \langle x_0,y_0\rangle & \|y_0\|^2 \end{pmatrix} \to \begin{pmatrix} \frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}$ and the ratios $\frac{\|x_0\|^2}{\|y_0\|^4}$, $\frac{\langle x_0,y_0\rangle}{\|y_0\|^3}$ tend to the same limits $\frac{1}{4}, -\frac{1}{2}$ as above.

For $\bar\tau \to 1$ we obtain

$$\begin{pmatrix} \|x_0\|^2 & \langle x_0,y_0\rangle \\ \langle x_0,y_0\rangle & \|y_0\|^2 \end{pmatrix} \to \begin{pmatrix} \frac{1}{4}\frac{(\tau_0(|\tau_0|-2)+1)^2}{\alpha^4} & -\frac{1}{2}\frac{(1-|\tau_0|)(\tau_0(|\tau_0|-2)+1)}{\alpha^3} \\ -\frac{1}{2}\frac{(1-|\tau_0|)(\tau_0(|\tau_0|-2)+1)}{\alpha^3} & \frac{(1-|\tau_0|)^2}{\alpha^2} \end{pmatrix}$$

and $\frac{\|x_0\|^2}{\|y_0\|^4} \to \frac{1}{4}\frac{(\tau_0(|\tau_0|-2)+1)^2}{(1-|\tau_0|)^4}$, $\frac{\langle x_0,y_0\rangle}{\|y_0\|^3} \to -\frac{1}{2}\frac{\tau_0(|\tau_0|-2)+1}{|1-|\tau_0||(1-|\tau_0|)}$. If $\tau_0 \geqslant 0$, then these limits are again equal to $\frac{1}{4}, -\frac{1}{2}$. For $\tau_0 \leqslant 0$ they equal $\frac{\left((1+\tau_0)^2-2\right)^2}{4(1+\tau_0)^4}$, $\frac{(1+\tau_0)^2-2}{2|1+\tau_0|(1+\tau_0)}$, and $\lim_{\bar\tau\to 1}\frac{\|x_0\|^2}{\|y_0\|^4} = \left(\lim_{\bar\tau\to 1}\frac{\langle x_0,y_0\rangle}{\|y_0\|^3}\right)^2$.

For $\bar\tau = 1$, $\tau_0 \in (-\infty,-1)$ the ratio $\frac{\langle x_0,y_0\rangle}{\|y_0\|^3}$ rises monotonely from $-\frac{1}{2}$ to $+\infty$. For $\tau_0 \in (-1,0]$ it rises monotonely from $-\infty$ to $-\frac{1}{2}$. The boundary values of the ratios $\frac{\|x_0\|^2}{\|y_0\|^4}$, $\frac{\langle x_0,y_0\rangle}{\|y_0\|^3}$ are depicted on Fig. 2.

Finally, if $\tau_0 \to -\infty$, then the Gramian grows unbounded. However, if we simultaneously let $\alpha \to +\infty$ such that $\frac{-\tau_0}{\alpha} \to 1$, then the leading terms in τ_0 dominate and again $\begin{pmatrix} \|x_0\|^2 & \langle x_0,y_0\rangle \\ \langle x_0,y_0\rangle & \|y_0\|^2 \end{pmatrix} \to \begin{pmatrix} \frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}$.

Hence if the parameters $\bar\tau,\tau_0$ circumvent the boundary of their domain of definition, the pair $(\frac{\|x_0\|^2}{\|y_0\|^4}, \frac{\langle x_0,y_0\rangle}{\|y_0\|^3})$ moves along the parabola on Fig. 2, including the infinitely far point. Except the interval $(\bar\tau,\tau_0) \in \{1\}\times\mathbb{R}_{--}$ the ratio pair tends to the point $(\frac{1}{4},-\frac{1}{2})$. For $\bar\tau \in (-1,+1)$, $\tau_0 < \bar\tau$ the pair takes values right of the parabola.

The values of $\alpha,\bar\tau,\tau_0$ producing a given Gramian of x_0,y_0 can then be obtained as follows. Compute the ratios $\frac{\|x_0\|^2}{\|y_0\|^4}$, $\frac{\langle x_0,y_0\rangle}{\|y_0\|^3}$. Determine the values of $\tau_0,\bar\tau$ yielding these ratios. This can be done, e.g., by tracing the level lines of the ratios as a function of $\bar\tau,\tau_0$, finding their intersection, and refining the values with the Newton method. In a final step determine α, e.g., from (18).

3. UPPER BOUND ON THE OBJECTIVE VALUE

In this section we compute the objective value of problem (5) on the time-optimal trajectory computed in Section 2. We again consider the two cases from the previous section.

Case $\alpha > 0$, $\beta > 0$: The objective value of the time-optimal trajectory for the original cost function is given by

$$\frac{1}{2}\int_{t_0}^{T}\|x(s)\|^2\,ds = \frac{1}{2}\int_{\tau_0}^{\bar\tau}\|x(\tau/\alpha)\|^2\,\frac{d\tau}{\alpha} = \frac{1}{2\alpha^5}\int_{\tau_0}^{\bar\tau}\alpha^4\|x\|^2\,d\tau.$$

Integrating expression (16) with respect to τ we get

$$\begin{aligned}
&\frac{\tau^5}{20} + \frac{1}{3}\left(\frac{71}{36}\beta^2 + \beta^2\operatorname{artanh}^2\bar\tau + 1\right)\tau^3 - \frac{1}{2}\left(\beta^2\operatorname{artanh}\bar\tau + \bar\tau\right)\tau^2 \\
&+ \frac{1}{36}\left(9\beta^4\operatorname{artanh}^2\bar\tau - 18\beta^2\bar\tau\operatorname{artanh}\bar\tau + 56\beta^4 + 27\beta^2 + 9\right)\tau \\
&- \frac{1}{9}\left(3\beta^2\bar\tau - 5\beta^4\operatorname{artanh}\bar\tau\right)\sqrt{\tau^2+\beta^2} - \frac{11\beta^2}{8}\tau\sqrt{\tau^2+\beta^2} \\
&+ \frac{1}{18}\left(13\beta^2\operatorname{artanh}\bar\tau + 3\bar\tau\right)\tau^2\sqrt{\tau^2+\beta^2} - \frac{1}{4}\tau^3\sqrt{\tau^2+\beta^2} \\
&- \frac{5\beta^4}{8}\operatorname{arsinh}\frac{\tau}{\beta} - \frac{\beta^2}{2}\left(\beta^2\operatorname{artanh}\bar\tau - \bar\tau\right)\tau\operatorname{arsinh}\frac{\tau}{\beta} + \frac{\beta^2}{2}\tau^2\operatorname{arsinh}\frac{\tau}{\beta} - \frac{2\beta^2}{3}\tau^3\operatorname{artanh}\bar\tau\operatorname{arsinh}\frac{\tau}{\beta} \\
&- \frac{5\beta^4}{9}\sqrt{\tau^2+\beta^2}\operatorname{arsinh}\frac{\tau}{\beta} - \frac{13\beta^2}{18}\tau^2\sqrt{\tau^2+\beta^2}\operatorname{arsinh}\frac{\tau}{\beta} + \frac{\beta^4}{4}\tau\operatorname{arsinh}^2\frac{\tau}{\beta} + \frac{\beta^2}{3}\tau^3\operatorname{arsinh}^2\frac{\tau}{\beta}.
\end{aligned}$$

For $\tau = \bar\tau$ this expression evaluates to

$$\frac{1}{1080}\left((1024\beta^4 - 163\beta^2 + 54)\bar\tau - 675\beta^4\operatorname{artanh}\bar\tau\right).$$

Hence the objective value $-\omega^{TO}$ of the time-optimal trajectory obeys

$$\begin{aligned}
-\alpha^5\omega^{TO} &= -\frac{\tau_0^5}{40} - \frac{1}{6}\left(\frac{71}{36}\beta^2 + \beta^2 \operatorname{artanh}^2\bar\tau + 1\right)\tau_0^3 + \frac{1}{4}(\beta^2 \operatorname{artanh}\bar\tau + \bar\tau)\tau_0^2 \\
&- \frac{1}{72}(9\beta^4 \operatorname{artanh}^2\bar\tau + 27\beta^2 + 56\beta^4 - 18\beta^2\bar\tau \operatorname{artanh}\bar\tau + 9)\tau_0 \\
&+ \frac{1}{2160}\left((1024\beta^4 - 163\beta^2 + 54)\bar\tau - 675\beta^4 \operatorname{artanh}\bar\tau\right) \\
&+ \frac{1}{18}(3\beta^2\bar\tau - 5\beta^4 \operatorname{artanh}\bar\tau)\sqrt{\tau_0^2+\beta^2} + \frac{11\beta^2}{16}\tau_0\sqrt{\tau_0^2+\beta^2} \\
&- \frac{1}{36}(13\beta^2 \operatorname{artanh}\bar\tau + 3\bar\tau)\tau_0^2\sqrt{\tau_0^2+\beta^2} + \frac{1}{8}\tau_0^3\sqrt{\tau_0^2+\beta^2} \\
&+ \frac{5\beta^4}{16}\operatorname{arsinh}\frac{\tau_0}{\beta} + \frac{\beta^2}{4}(\beta^2 \operatorname{artanh}\bar\tau - \bar\tau)\tau_0 \operatorname{arsinh}\frac{\tau_0}{\beta} - \frac{\beta^2}{4}\tau_0^2 \operatorname{arsinh}\frac{\tau_0}{\beta} \\
&+ \frac{\beta^2}{3}\tau_0^3 \operatorname{artanh}\bar\tau \operatorname{arsinh}\frac{\tau_0}{\beta} + \frac{5\beta^4}{18}\sqrt{\tau_0^2+\beta^2}\operatorname{arsinh}\frac{\tau_0}{\beta} + \frac{13\beta^2}{36}\tau_0^2\sqrt{\tau_0^2+\beta^2}\operatorname{arsinh}\frac{\tau_0}{\beta} \\
&- \frac{\beta^4}{8}\tau_0 \operatorname{arsinh}^2\frac{\tau_0}{\beta} - \frac{\beta^2}{6}\tau_0^3 \operatorname{arsinh}^2\frac{\tau_0}{\beta}.
\end{aligned}$$

Case $\alpha\beta = 0$: For the 1-dimensional control problem we have

$$\frac{x(t)^2}{2} = \begin{cases} \dfrac{t^4}{8}, & \text{if } t > -\sqrt{\sigma x_0 + \dfrac{y_0^2}{2}}, \\ \dfrac{1}{2}\left(\dfrac{t^2}{2} + 2\sqrt{\sigma x_0 + \dfrac{y_0^2}{2}}\,t + \sigma x_0 + \dfrac{y_0^2}{2}\right)^2, & t < -\sqrt{\sigma x_0 + \dfrac{y_0^2}{2}}. \end{cases}$$

Hence with $\tilde\phi^{-1} = -\sqrt{\sigma x_0 + \frac{y_0^2}{2}}$, $\sigma = \operatorname{sgn}\left(x_0 + \frac{y_0|y_0|}{2}\right)$, and $t_0 = -\sigma y_0 + 2\tilde\phi^{-1}$ the objective value of problem (1) on the time-optimal trajectory is given by

$$\begin{aligned}
\frac{1}{2}\int_{t_0}^{0} x(t)^2\,dt &= \frac{1}{2}\int_{t_0}^{\tilde\phi^{-1}}\left(\frac{t^2}{2} - 2\tilde\phi^{-1}t + \tilde\phi^{-2}\right)^2 dt + \int_{\tilde\phi^{-1}}^{0}\frac{t^4}{8}\,dt \\
&= -\frac{t_0^5}{40} + \frac{t_0^4\tilde\phi^{-1}}{4} - \frac{5}{6}t_0^3\tilde\phi^{-2} + t_0^2\tilde\phi^{-3} - \frac{t_0\tilde\phi^{-4}}{2} + \frac{\tilde\phi^{-5}}{12} \\
&= -\frac{23}{60}\tilde\phi^{-5} + \frac{\tilde\phi^{-4}\sigma y_0}{2} - \frac{\tilde\phi^{-2}\sigma y_0^3}{6} + \frac{\sigma y_0^5}{40} \\
&= \frac{\sigma y_0 x_0^2}{2} + \frac{x_0 y_0^3}{3} + \frac{\sigma y_0^5}{15} + \frac{23}{60}\left(\sigma x_0 + \frac{y_0^2}{2}\right)^{5/2}. \qquad (19)
\end{aligned}$$

4. QUALITY OF APPROXIMATION

In this section we compare the objective value of the constructed sub-optimal solution with the optimal objective value on those trajectories where the latter is known.

Let us first consider the 1-dimensional problem (1). Since both the optimal value (2) (multiplied by -1) and the objective value (19) of the time-optimal trajectory satisfy the symmetry (4), the relative gap between the two values depends only on the ratio $\frac{x}{|y|y}$. This gap is depicted on Fig. 3. It varies between approximately 5.4×10^{-5} and 5.6×10^{-2}.

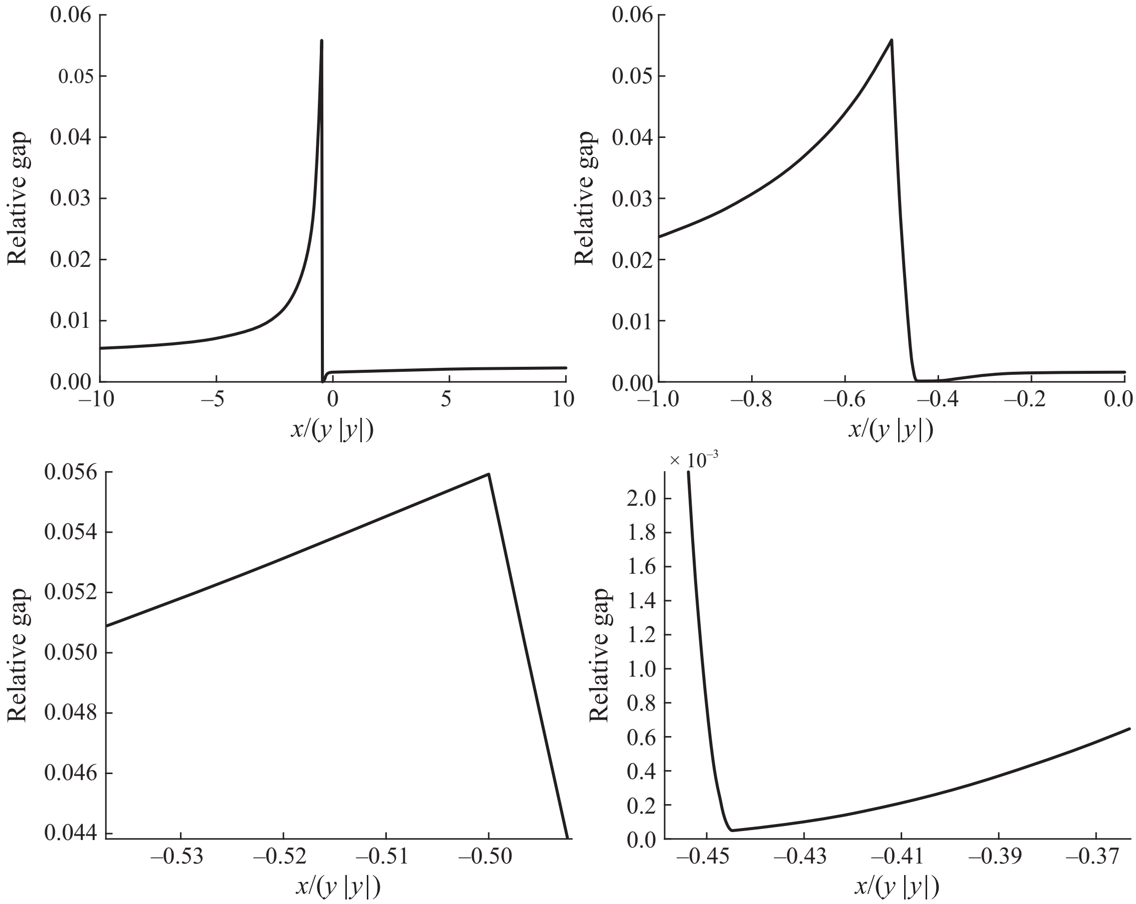

Fig. 3. Relative gap between the value of objective (1) on the time-optimal trajectory and on the optimal solution. The time-optimal trajectory switches control on the curve $x = -\frac{y|y|}{2}$, whereas the optimal trajectory switches control on the curve $x = -\beta y|y|$ with $\beta \approx 0.4446$. The figures on the right and bottom are zooms of the upper left figure.

Since both the time-optimal problem (10) and problem (5) reduce to their 1-dimensional versions Fig. 1 and (1) if the initial values of the vectors x, y are collinear, the same gap is achieved for the 2-dimensional problems for these initial values.

Let us now consider the self-similar trajectories found in [8]. First we compute the optimal value of problem (5) on these trajectories. By virtue of (4) the Bellman function on the self-similar trajectories satisfying (8) obeys

$$\omega_{2D}(x(t), y(t)) = \frac{\lambda(t)^5}{\lambda_0^5}\omega_{2D}(x(0), y(0)).$$

Differentiating with respect to t and using $\frac{d\omega(x(t),y(t))}{dt} = \frac{1}{2}\|x(t)\|^2$ yields

$$\omega_{2D}(x(0), y(0)) = -\frac{\lambda_0^5}{540}. \tag{20}$$

We now consider the value of the objective on the time-optimal trajectory with the same initial values (7). To this end we have to invert relations (16), (17), (18), i.e., determine the values of $\alpha, \tau, \bar{\tau}$ yielding these initial values.

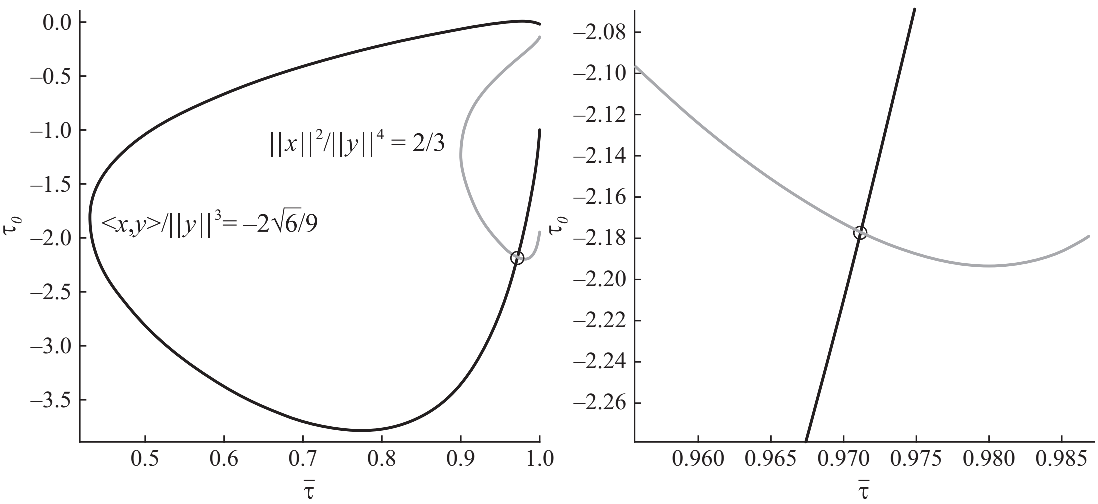

Fig. 4. Level curves of the ratios $\frac{\|x_0\|^2}{\|y_0\|^4}$, $\frac{\langle x_0,y_0\rangle}{\|y_0\|^3}$ corresponding to the self-similar trajectory in the $(\bar\tau,\tau_0)$ plane. The values of $\bar\tau,\tau_0$ producing an initial point on this trajectory are given by the unique intersection point of the curves (circle).

Following the scheme outlined in Section 2, we first consider the level curves of the ratios $\frac{\|x_0\|^2}{\|y_0\|^4}$, $\frac{\langle x_0,y_0\rangle}{\|y_0\|^3}$ in the $(\bar\tau,\tau_0)$ plane. From (7) we get the values

$$\frac{\|x_0\|^2}{\|y_0\|^4}=\frac{2}{3},\quad \frac{\langle x_0,y_0\rangle}{\|y_0\|^3}=\frac{2\sqrt{6}}{9}.$$

The corresponding level curves are depicted on Fig. 4. Refining the values obtained graphically by a Newton method we get

$$\bar\tau\approx 0.97116420999,\quad \tau_0\approx -2.17695799429.$$

Inserting into (18) and setting $\|y_0\|^2=\frac{\lambda_0^2}{6}$ by virtue of (7), we further obtain the value $\alpha\approx 4.13415835032\lambda_0^{-1}$.

Inserting the parameter values into the expression for the objective value on the time-optimal trajectory, we obtain the sub-optimal cost $\approx 0.0019779902706\lambda_0^5$. Compared with the optimal cost (20) computed above, this yields a relative gap of $\approx 6.8\times 10^{-2}$.

Let us now compare the optimal control on the self-similar trajectory with the time-optimal control on this trajectory. We choose the initial point which corresponds to the value $\lambda_0=1$, thus the optimal trajectory needs unit time to arrive at the origin.

The time-optimal control is given by (13), where τ runs from τ_0 to $\bar\tau$ and the time variable correspondingly from 0 to $T^{TO}=\alpha^{-1}(\bar\tau-\tau_0)\approx 0.7614904746$. Note that the control evolves clockwise around the origin. Note also that the arrival time at the origin is smaller than the arrival time $T=1$ for the optimal trajectory, because the time-optimal control minimizes precisely the arrival time. On Fig. 5 the polar angle of the time-optimal control is depicted as a function of time.

By (9) the optimal control on the self-similar trajectory evolves according to the formula

$$\hat u=(\cos\varphi(t),\sin\varphi(t))^T,\quad \varphi(t)=\sqrt{5}\log(T-t)+\text{const},$$

where $T=\lambda_0=1$ is the arrival time of the trajectory at the origin and the constant is the polar angle of the control at the initial time instant $t=0$. In order to determine this constant we first compute the initial value of y by formulas (14),(15). It amounts to

$$y(0)\approx\begin{pmatrix}0.2878394861\\ -0.2895083711\end{pmatrix}.$$

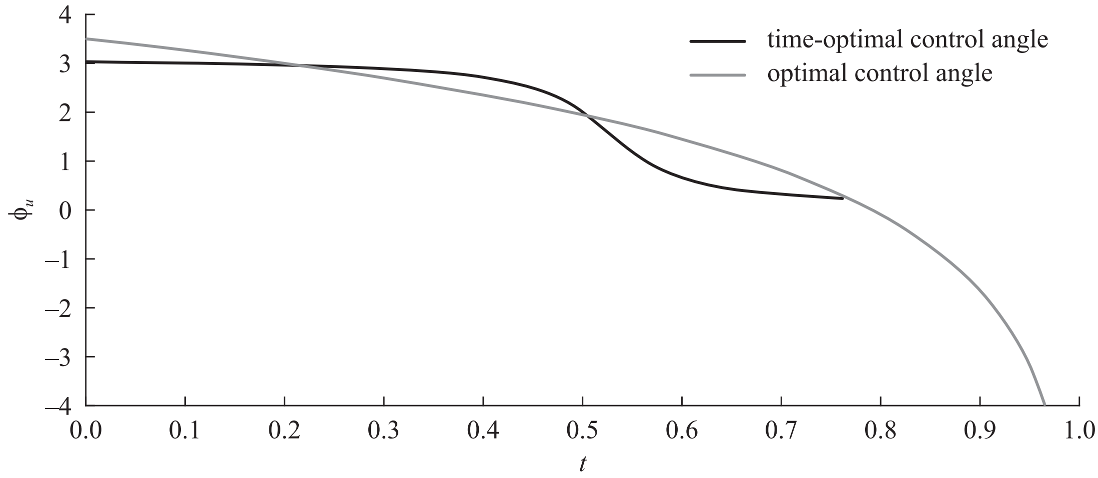

Fig. 5. Evolution of the polar angle of the optimal control and the time-optimal control for the same initial point, located on a self-similar trajectory with arrival time $T = 1$. For $t \to 1$ the optimal control angle tends to infinity.

As mentioned in Section 1, the optimal control is directed at an angle $\pi - \operatorname{atan}\sqrt{5}$ relative to y, leading to an initial control angle of ≈ 3.5035658841. For $t \to T = 1$ the polar angle of the control decreases logarithmically, and the control revolves an infinite number of times around the origin. The evolution of the angle as a function of time is depicted on Fig. 5.

5. CONCLUSIONS

In this paper we considered two optimal control problems, which share the feasible set of trajectories but have different objective values. While the time-optimal problem (10) can be solved analytically, for problem (5), which exhibits a singular trajectory of second order, only a limited number of optimal trajectories are known explicitly.

We describe the solution of the time-optimal control problem and use its solution to construct an upper bound on the objective value of problem (5). Comparison of the value of the sub-optimal (time-optimal) solution with the value of known optimal trajectories shows that the relative gap in objective value ranges from several thousandth of a per cent to several per cent. The difference in the polar angle of the two controls can, however, be quite substantial (up to 45 degrees).

The upper bound can be used to constrain the locus of the optimal trajectories of problem (5) in extended phase space (i.e., jointly with the adjoint variables) and thus simplify the analysis of the optimal synthesis of this problem. More concretely, it was shown in [9] that the Fuller symmetry (3) implies that the Bellman function is given by $\omega(x, y) = \frac{1}{5}(\langle\psi, y\rangle + 2\langle\phi, x\rangle)$, where ϕ, ψ are the adjoint variables to x, y. Hence an upper bound on the objective value at a given point (x, y) implies a linear inequality on the optimal values of the adjoint variables at this point.

Numerical experiments show that the self-similar trajectory is repulsive in the factor (orbit) space with respect to the action of the symmetry groups, while the trajectories corresponding to the 1D analog (1) are attractive. A rigorous proof of this property and a qualitative description of the complete optimal synthesis of the problem remain open and will be subject of future investigations.

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